NOTATION

U) voltage; I) current; ρ) specific electrical resistance; T) conductor temperature; d) conductor diameter; R) conductor resistance; ΔT) temperature head; q) heat flow density; α) heat-transfer coefficient; l) conductor length; Nu) Nusselt number; Ra) Rayleigh number; Θ) Debye characteristic temperature; T_{mp}) fusing temperature; γ) density; g) acceleration of gravity; C) specific heat; β) volumetric expansion coefficient; η) dynamic viscosity; λ) coefficient of thermal conductivity. Indices: 0) temperature of heat carrier; max and min) maximal and minimal values; int) integral with respect to the conductor's length.

LITERATURE CITED

1. A. A. Slutskin and A. M. Kadigrobov, Pis'ma Zh. Éksp. Teor. Fiz., 28, No. 4, 219-222 (1978).
2. V. V. Barelko, V. M. Beibutyan, Yu. E. Volodin, et al., Dokl. Akad. Nauk SSSR, 257, No. 2, 339-344 (1981).
3. V. V. Boiko, Yu. F. Podrezov, and N. P. Klimova, Pis'ma Zh. Éksp. Teor. Fiz., 35, No. 12, 524-526 91982).
4. G. I. Abramov, A. V. Gurevich, V. M. Dzugutov, et al., Pis'ma Zh. Éksp. Teor. Fiz., 37, No. 10, 453-456 (1983).
5. G. I. Abramov, A. V. Gurevich, S. I. Zakharchenko, et al., Fiz. Tverd. Tela, 27, No. 8, 2250-2257 (1985).
6. S. A. Zhukov and B. B. Barelko, Int. J. Heat Mass Transfer, 26, No. 8, 1121-1130 (1983).
7. G. I. Abramov, S. I. Zakharchenko, and L. M. Fisher, Teplofiz. Vys. Temp., 24, No. 4, 736-742 (1986).
8. G. I. Abramov, Proceedings of MÉI, No. 427, 97-104 (1979).
9. G. I. Abramov, Proceedings of MÉI, No. 427, 104-113 (1979).
10. G. I. Abramov, Teploénergetika, No. 9, 55-57 (1981).
11. G. I. Abramov, Teploénergetika, No. 12, 60-62 (1982).
12. V. P. Isachenko, V. A. Osipova, and A. S. Sukomel, Heat Transfer [in Russian], Moscow (1980).
13. G. I. Abramov and V. V. Kuznetsov, Inzh.-fiz. Zh., 56, No. 5, 859-860 (1989).

NUMERICAL INVERSION OF A LAPLACE TRANSFORM USING A FOURIER SERIES TO COMPUTE NONSTATIONARY TEMPERATURE FIELDS IN LAYERED STRUCTURES

O. V. Kanunnikov and S. L. Ésaulov

UDC 536.21:550.362

An algorithm is examined for the selection and assignment of the parameter σ of a numerical inversion of a Laplace transform with the use of Fourier series.

In the field of structural thermal physics, significant attention is allotted to questions of heat- and mass-transport in layered structures. This interest in multilayered systems is explained by the fact that they permit one to more rationally exploit the thermophysical and physical–mechanical properties of building materials.

Examined below is the problem of determining nonstationary temperature fields in multilayered building structures, situated on the ground half-space. In the general case, they are represented as a system of infinite plates with internal heat sources and sinks. Ideal contact is maintained between the layers of the structure and the ground mass, i.e., fourth-order boundary conditions are realized. The thermophysical characteristics of the materials in the layers are different. The temperature of the medium varies harmonically. The heat transfer conditions between the medium and the surface of the structure are subject to Newton's

Translated from Inzhernerno-fizicheskii Zhurnal, Vol. 61, No. 5, pp. 804-807, November, 1991. Original article submitted February 22, 1991.

 0022-0841/91/6105-1390$12.50

law. With these preconditions and assumptions, the system of differential equations for the problem just formulated has the form [1]

$$\frac{\partial T_i(x,\ \tau)}{\partial \tau} = a_i \frac{\partial^2 T_i(x,\ \tau)}{\partial x^2} + \frac{\omega_i}{c_i \gamma_i}. \tag{1}$$

The initial conditions are:

at $\tau = 0$

$$T_i(x,\ 0) = u_i(x). \tag{2}$$

The boundary conditions are:

at $x = 0$

$$\alpha\,[T_1(0,\ \tau) - T_c(\tau)] = \lambda_1 \frac{\partial T_1(0,\ \tau)}{\partial x}; \tag{3}$$

at $x = h_i$

$$T_i(h_i,\ \tau) = T_{i+1}(h_i,\ \tau); \tag{4}$$

$$\lambda_i \frac{\partial T_i(h_i,\ \tau)}{\partial x} = \lambda_{i+1} \frac{\partial T_{i+1}(h_i,\ \tau)}{\partial x}; \tag{5}$$

at $x = h_n$

$$\frac{\partial T_n(h_n,\ \tau)}{\partial x} = 0, \tag{6}$$

where $i = 1, 2, 3, \ldots, n$.

After applying an integral Laplace transform with respect to the variable τ to the system of differential equations (1) and the corresponding initial condition (2) and boundary conditions (3)-(6), we obtain a solution to the problem in transforms of the temperature functions in the individual layers [2]:

$$\tilde{T}_i(x,\ s) = A_i \operatorname{ch} \sqrt{\frac{s}{a_i}}\, x + B_i \operatorname{sh} \sqrt{\frac{s}{a_i}}\, x + \frac{\omega_i}{s c_i \gamma_i} - \frac{1}{\sqrt{a_i s}} \int_0^x u_i(\xi) \operatorname{sh} \sqrt{\frac{s}{a_i}}\,(x - \xi)\, d\xi. \tag{7}$$

In this problem, the derivation of closed analytical expressions for the desired temperature functions both for transforms and for inverse transforms for two or more layers presents significant difficulty; therefore, the transition from transforms to inverse transforms is done numerically.

At the present time, several practical means to numerically invert a Laplace transform have been developed. These methods are based on determining the numerical values of the inverse transform from the corresponding values of the transforms at equidistant points on the real axis [3, 4]. To solve the problem, the method of numerical inversion of a Laplace transform via Fourier series is used [5]. The essence of this approach consists in the fact that the familiar Laplace integral

$$\tilde{T}_i(x,\ s) = \int_0^\infty T_i(x,\ \tau) \exp(-s\tau)\, d\tau \tag{8}$$

is transformed via the substitution

$$\exp(-\sigma\tau) = \cos\theta, \tag{9}$$

where σ is an arbitrary positive number. Then

$$T_i(x,\ \tau) = T_i\left(x,\ -\frac{1}{\sigma} \ln \cos\theta\right) = T_i(x,\ \theta), \tag{10}$$

$$\sigma \tilde{T}_i(x,\ s) = \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{s}{\sigma} - 1} \sin\theta\, T_i(x,\ \theta)\, d\theta. \tag{11}$$

Performing a change of variable allows the function $T_i(x, \Theta)$ to be decomposed into a Fourier series in sines:

$$T_i(x, \theta) = \sum_{m=0}^{\infty} C_{im} \sin(2m+1)\theta, \tag{12}$$

where C_{im}, is determined from a system of recurrence relations.

The main interest when using this method in problems of calculating nonstationary temperature fields in semifinite layered systems is the determination and selection of the positive parameter σ. In engineering practice, as a rule, it is necessary to know the temperature fields in the construction elements over certain time intervals. To do this, it is necessary to determine $T_i(x, \theta)$ for the values

$$\theta = \arccos[\exp(-\sigma\tau)]. \tag{13}$$

The numerical value of arbitrary positive σ depends on the size of the interval inside of which it is necessary to calculate the inverse transform function. Analysis of domestic and foreign research into this question has shown that there is no universal algorithm for calculating σ [6]. This limits the use of the proposed means to numerically invert the Laplace transform. At the same time, correctly selecting the value of σ is an independent problem, which determines the asymptotic convergence of the solution as $\tau \to \tau_k$, where τ_k is the point in time for which it is necessary to obtain the numerical value of the inverse transform.

The approach proposed in this article for determining the parameter σ consists in estimating the total error of inverting the Laplace transform via a Fourier series as compared to an etalon-function, the analytic form of whose Laplace transform is known and similar in form to the function causing difficulty [7]. In a certain sense, the etalon-function is calculated by a model of some ideal process, which reflects only the basic properties of the structure and phenomena occurring in it.

It should be kept in mind that the transform function for each layer of the system being examined can be represented in generalized form as $F_1(x, s)$, $F_2(x, s)$, ..., $F_n(x, s)$. On this basis, and also from previous research of the authors, the positive parameter σ will vary from layer to layer.

It is assumed that the set of $\sigma_{k,i}$, where $i = 1, \ldots, n$, obtained as a result of numerical experiment, can be used to help solve similar problems of this class. For the studied class of problems, the known solution for calculating temperature fields in semi-finite homogeneous massif can be used in the capacity of an etalon-function [1].

The method of determining and assigning the numerical values of the parameter σ consists of the following:

1. The moment of time τ_k is fixed.

2. Considering that $0 < \theta < 90°$, $\theta_{min} = 0.0001°$ and $\theta_{max} = 89.0000°$ are given, and

$$\sigma_{max} = -\frac{\ln(\cos\theta_{max})}{\tau_k}; \quad \sigma_{min} = -\frac{\ln(\cos\theta_{min})}{\tau_k}; \quad \Delta\sigma = \frac{\sigma_{max} - \sigma_{min}}{100}$$

are correspondingly determined.

3. By sequentially decreasing σ from the value σ_{max} by an amount $\Delta\sigma$, we determine σ^*, for which the calculated values of the inverse transform function approach some constant values that differ from zero (nonsingular stable solution in the sense of [4]).

4. On the segment $[\sigma_{min}, \sigma^*]$, by sequentially varying the value of σ, we choose $\sigma_{k,i}$ for which the total error of the inversion at the surface of the i-th layer is no greater than δ_{max}, and at the bottom of the i-th layer is no less than $-\delta_{max}$. The calculation is performed, starting with $i = 1$. For this value of i, the problem is solved in the total volume for n layers, with combined second and third order boundary conditions at the surface of the first layer and with second order boundary conditions at the bottom of the lower layer. For $i \geq 2$, the problem is solved only for $(n - i + 1)$ lower layers; the first order boundary conditions calculated in the previous step are applied to the surface of the i-th layer.

5. The procedure in paragraph 4 is performed sequentially for all layers. The set of $\sigma_{k,i}$ is obtained as a result, where $i = 1, \ldots, n$.

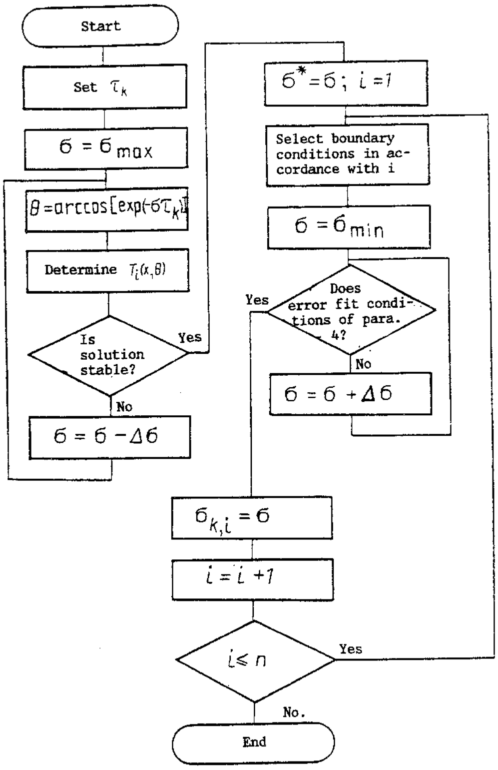

Fig. 1. Algorythm for selecting and assigning the parameter σ of the numerical inversion of a Laplace transform using a Fourier series.

6. Paragraphs 1-5 are performed for all calculated points in time τ_k.

The sequence of actions for determining and assigning the parameters $\sigma_{k,i}$ at the moment in time τ_k is shown in Fig. 1 as a generalized algorithm.

The studies conducted by the authors have shown that the error in determining the values of the inverse transform functions when using this method to numerically invert the Laplace transform is less than 5%. This kind of accuracy is acceptable for calculating nonstationary temperature fields in layered building structures.

NOTATION

σ) positive parameter; θ) auxiliary variable; x, ξ) coordinates, m; h_i) distance from the surface of the structure to the bottom of the i-th layer, m; τ, τ_k) time, h; $T_i(x, \tau)$) temperature of the i-th layer in the transform region; A_i, B_i, C_{im} are numerical coefficients; a_i is the temperature conductivity coefficient, m^2/h; ω_i is the power density of a heat source in the i-th layer, W/m^3; c_i is the specific heat capcity, kJ/kg·°C; λ_i is the thermal conductivity coefficient, W/m·°C; γ_i is the density, kg/m^3; α is the coefficient of heat transfer of the surface of the structure, $W/m^2\cdot°C$; δ_{max} is the maximum permissible error, %; $T_c(\tau)$ is the temperature of the medium, °C.

LITERATURE CITED

1. A. V. Lykov, Analytical Heat Diffusion Theory, Academic Press, New York (1968).
2. A. V. Lykov, Thermal Conductivity of Nonstationary Processes [in Russian], Moscow (1948).
3. C. Lanczos, Applied Analysis, Prentice-Hall, Englewood Cliffs, NJ (1956).
4. G. Doetsch, Guide to the Applications of the Laplace and Z-Transforms, Van Nostrand–Reinhold, New York (1971).
5. V. I. Krylov and N. S. Skoblya, Handbook on Numerical Inversion of Laplace Transforms, Israel Program for Scientific Translations, Jerusalem (1969).
6. A. Papoulis, Quart. Appl. Math., 14, No. 4, 405-411 (1957).
7. B. Davies and B. Martin, J. Comput. Phys., 33, No. 1, 1-32 (1979).